INVERSE PROBLEM OF FINDING THE TIME-DEPENDENT COEFFICIENT OF HEAT EQUATION FROM INTEGRAL OVERDETERMINATION CONDITION DATA

MANSUR I. ISMAILOV*) FATMA KANCA**)

ABSTRACT. In this paper we consider the problem of simultaneously determining the time-dependent thermal diffusivity and the temperature distribution in one-dimensional heat equation in the case of nonlocal boundary and integral overdetermination conditions. We establish conditions for the existence and uniqueness of a classical solution of the problem under considerations. We present some results on the numerical solution with an example.

INTRODUCTION

Suppose that one need to determine the temperature distribution $u(x,t)$ as well as thermal coefficient $a(t)$ simultaneously satisfy the equation

$$(0.1) \qquad u_t = a(t)u_{xx} + F(x,t), \ 0 < x < 1, \ 0 < t \le T,$$

with the initial condition

$$(0.2) \qquad u(x,0) = \varphi(x), \ 0 \le x \le 1,$$

the boundary conditions

$$(0.3) \qquad u(0,t) = u(1,t), \ u_x(1,t) = 0, \ 0 \le t \le T,$$

and the overdetermination condition

$$(0.4) \qquad \int_0^1 u(x,t)dx = E(t), \ 0 \le t \le T.$$

The problem of finding the pair $\{a(t), \ u(x,t)\}$ in (0.1)-(0.4) will be called an inverse problem.

Denote the domain Q_T by

$$Q_T = \{(x,t): \ 0 < x < 1, \ 0 < t \le T\}.$$

Definition 1. *The pair $\{a(t),\ u(x,t)\}$ from the class $C\,[0,T]{\times}C^{2,1}\,(Q_T){\cap}C^{1,0}\left(\overline{Q}_T\right)$ for which conditions (0.1)-(0.4) are satisfied and $a(t)>0$ on the interval $[0,T]$, is called the classical solution of the inverse problem (0.1)-(0.4).*

The parameter identification in a parabolic differential equation from the data of integral overdetermination condition plays an important role in engineering and physics. ([1, 2, 3, 4, 5])

Various statements of inverse problems on determination of thermal coefficient in one-dimensional heat equation were studied in [4,5,6]. It is important to note that in the papers [4,5] the time dependent thermal coefficient is determined by nonlocal overdetermination condition's data. Besides, in [1,4] the coefficients of the heat equations are determined in the case of nonlocal boundary conditions.

In the present work, the existence and uniqueness of the classical solution of the problem (0.1)-(0.4) is reduced to fixed point principles by applying Fourier method. The boundary conditions (0.3) admit the expansions by the system of eigenfunctions and associated functions corresponding to the spectral problem.

The paper organized as follows:

In Chapter 1, the auxiliary spectral problem which can be obtained by applying Fourier method to the problem (0.1)-(0.3) is studied. In Chapter 2, the existence of the solution of inverse problem in Q_T, and the uniqueness of the solution of the inverse problem in Q_{T_0} $(0<T_0\le T)$ are shown. Then in Chapter 3, the continuous dependence upon the solution of the inverse problem is shown. Finally, in Chapter 4, the numerical solution for the inverse problem is presented with an example.

1. The auxiliary spectral problem

Consider the spectral problem

$$X''(x)+\lambda X(x)=0,\ 0\le x\le 1,$$

$$(1.1)\qquad\qquad\qquad X(0)=X(1),\ X'(1)=0.$$

This problem is wellknown in [7], as auxiliary spectral problem in solving a nonlocal boundary value problem for heat equation by Fourier method.

It is clear to show that, the problem (1.1) has eigenvalues

$$\lambda_k=(2\pi k)^2, k=0,1,2,...$$

and eigenfunctions

$$(1.2)\qquad\qquad \overline{X}_0(x)=2,\ \overline{X}_k(x)=4\cos 2\pi kx,\ k=1,2,...$$

and the system of functions $\overline{X}_k(x),\ k=0,1,2,...$ is not basis in $L_2[0,1]$. Complete the system $\overline{X}_k(x),\ k=0,1,2,...$ with the associated functions

$$(1.3)\qquad\qquad \overline{\overline{X}}_k(x)=4(1-x)\sin 2\pi kx, k=1,2,...$$

of the problem (1.1). Denote the systems of functions (1.2) and (1.3) as follows:

$$(1.4)$$
$$X_0(x)=2,\ X_{2k-1}(x)=4\cos 2\pi kx,\ X_{2k}(x)=4(1-x)\sin 2\pi kx,\ k=1,2,... \ .$$

The system of functions $X_k(x)$, $k = 0, 1, 2, ...$ is basis in $L_2[0, 1]$. ([8])

The adjoint problem of (1.1) has the form

$$Y''(x) + \lambda Y(x) = 0, \ 0 \leq x \leq 1,$$

(1.5) $$Y(0) = 0, \ Y'(0) = Y'(1).$$

Analogously to the system (1.4), the system of eigenfunctions and associated functions of the problem (1.5) is denoted by

(1.6) $$Y_0(x) = x, \ Y_{2k-1}(x) = x \cos 2\pi kx, \ Y_{2k}(x) = \sin 2\pi kx, \ k = 1, 2, ...$$

It is easy to calculate that the systems (1.4) and (1.6) form a biorthonormal system on interval $[0, 1]$, i.e.

$$(X_i, Y_j) = \int_0^1 X_i(x) Y_j(x) dx = \delta_{ij} = \left\{ \begin{array}{l} 0, i \neq j \\ 1, i = j \end{array} \right. .$$

2. EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE INVERSE PROBLEM

We have the following assumptions on the data of the problem (0.1)-(0.4).

(A_1) $E(t) \in C^1[0, T]$, $E'(t) < 0$, $\forall t \in [0, T]$;

(A_2) $\varphi(x) \in C^4[0, 1]$;

(1) $\varphi(0) = \varphi(1)$, $\varphi'(1) = 0$, $\varphi''(0) = \varphi''(1)$, $\int_0^1 \varphi(x) dx = E(0)$;

(2) $\varphi_{2k} \geq 0$, $k = 1, 2, ...$;

(A_3) $F(x, t) \in C(\overline{Q}_T)$; $F(x, t) \in C^4[0, 1]$ for arbitrary fixed $t \in [0, T]$;

(1) $F(0, t) = F(1, t)$, $F_x(1, t) = 0$, $F_{xx}(0, t) = F_{xx}(1, t)$;

(2) $F_{2k}(t) \geq 0$, $k = 0, 1, 2, ...$, $\int_0^T E'(t) dt + \sum_{k=1}^{\infty} \frac{2}{\pi k} \varphi_{2k} - 2 \int_0^T F_0(t) dt > 0$,

where $\varphi_k = \int_0^1 \varphi(x) Y_k(x) dx$, $F_k(t) = \int_0^1 F(x, t) Y_k(x) dx$, $k = 0, 1, 2, ...$.

Remark 1. *There are functions φ, E and F satisfying the assumptions $(A_1) - (A_3)$. For example*

$\varphi(x) = (1 - x) \sin 2\pi x$,

$E(t) = \frac{1}{2\pi} \exp(-t)$,

$F(x, t) = (1 - x) \sin 2\pi x \exp(3t)$.

The main result is presented as follows.

Theorem 1. *Let the assumptions $(A_1) - (A_3)$ be satisfied. Then the following statements are true:*

(1) *The inverse problem (0.1)-(0.4) has a solution in Q_T;*

(2) *The solution of inverse problem (0.1)-(0.4) is unique in Q_{T_0}, where the number T_0 $(0 < T_0 < T)$ is determined by the data of the problem.*

Proof. Any solution of the equation (0.1) can be given by

$$(2.1) \qquad u(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k(x),$$

where the functions $u_k(t)$, $k = 0, 1, 2, \dots$ satisfy the following system of equations:

$$u_0'(t) = F_0(t),$$

$$u_{2k}'(t) + (2\pi k)^2 a(t) u_{2k}(t) = F_{2k}(t),$$

$$u_{2k-1}'(t) + (2\pi k)^2 a(t) u_{2k-1}(t) + 4\pi k u_{2k}(t) = F_{2k-1}(t), \quad k = 1, 2, \dots .$$

Substuting the solution of this system of equations and initial condition (0.2) in (2.1), we obtain the solution of the problem (0.1)-(0.3) in the following form

$$
\begin{aligned}
u(x,t) \;=\; & \left[\varphi_0 + \int_0^t F_0(\tau)d\tau \right] X_0(x) \\
(2.2) \qquad & + \sum_{k=1}^{\infty} \left[\varphi_{2k} e^{-(2\pi k)^2 \int_0^t a(s)ds} + \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 \int_\tau^t a(s)ds} d\tau \right] X_{2k}(x) \\
& + \sum_{k=1}^{\infty} \left[\left(\varphi_{2k-1} - 4\pi k \varphi_{2k} t \right) e^{-(2\pi k)^2 \int_0^t a(s)ds} \right] X_{2k-1}(x) \\
& + \sum_{k=1}^{\infty} \left[\int_0^t \left(F_{2k-1}(\tau) - 4\pi k F_{2k}(\tau)(t-\tau) \right) e^{-(2\pi k)^2 \int_\tau^t a(s)ds} d\tau \right] X_{2k-1}(x).
\end{aligned}
$$

Under the conditions (1) of (A_2) and (1) of (A_3) the series (2.2) and $\sum_{k=1}^{\infty} \frac{\partial}{\partial x}$ are uniformly convergent in $\overline{Q}_T$ since their majorizing sums are absolutely convergent. Therefore their sums $u(x,t)$ and $u_x(x,t)$ are continuous in $\overline{Q}_T$. In addition, the series $\sum_{k=1}^{\infty} \frac{\partial}{\partial t}$ and $\sum_{k=1}^{\infty} \frac{\partial^2}{\partial x^2}$ are uniformly convergent in Q_T. Thus, we have $u(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(\overline{Q}_T)$. In addition, $u_t(x,t)$ is continuous in $\overline{Q}_T$. Differentiating (0.4) under the condition (A_1), we obtain

$$(2.3) \qquad \int_0^1 u_t(x,t)dx = E'(t), \; 0 \le t \le T,$$

and this yields

$$(2.4) \qquad a(t) = P\left[a(t)\right],$$

where

$$(2.5) \quad P\left[a(t)\right] = \frac{2F_0(t) + \sum\limits_{k=1}^{\infty} \frac{2}{\pi k} F_{2k}(t) - E'(t)}{\sum\limits_{k=1}^{\infty} 8\pi k \left(\varphi_{2k} e^{-(2\pi k)^2 \int\limits_0^t a(s)ds} + \int\limits_0^t F_{2k}(\tau) e^{-(2\pi k)^2 \int\limits_\tau^t a(s)ds} d\tau \right)}.$$

Denote

$$C_0 = 2 \min_{t \in [0,T]} F_0(t) + \min_{t \in [0,T]} \left(\sum_{k=1}^{\infty} \frac{2}{\pi k} F_{2k}(t) \right) - \max_{t \in [0,T]} E'(t),$$

$$C_1 = 2 \max_{t \in [0,T]} F_0(t) + \max_{t \in [0,T]} \left(\sum_{k=1}^{\infty} \frac{2}{\pi k} F_{2k}(t) \right) - \min_{t \in [0,T]} E'(t),$$

$$C_2 = \int_0^T E'(t)dt + \sum_{k=1}^{\infty} \frac{2}{\pi k} \varphi_{2k} - 2 \int_0^T F_0(t)dt,$$

$$C_3 = \sum_{k=1}^{\infty} 8\pi k \left(\varphi_{2k} + \int_0^T F_{2k}(\tau)d\tau \right).$$

It is easy to verify that $C_k > 0$, $k = 1, 2, 3, 4$ and $C_2 \leq C_3$, such that

$$\int_0^T E'(t)dt + \sum_{k=1}^{\infty} \frac{2}{\pi k} \varphi_{2k} - 2 \int_0^T F_0(t)dt \leq \sum_{k=1}^{\infty} \frac{2}{\pi k} \left(\varphi_{2k} e^{-(2\pi k)^2 \int\limits_0^T a(s)ds} + \int_0^T F_{2k}(\tau) e^{-(2\pi k)^2 \int\limits_\tau^T a(s)ds} d\tau \right).$$

Using the representation (2.4), the following estimate is true:

$$0 < \frac{C_0}{C_3} \leq a(t) \leq \frac{C_1}{C_2}.$$

Introduce the set M as:

$$M = \left\{ a(t) \in C\left[0, T\right] : \frac{C_0}{C_3} \leq a(t) \leq \frac{C_1}{C_2} \right\}.$$

It is easy to see that

$$P : M \to M.$$

Show that the operator P is compact. Let $M_1 \subset M$ be an arbitrary bounded set. Since $P(M_1) \subset M$, then $P(M_1)$ is uniformly bounded. Then, we have for $a(t) \in M_1$ and $t_1, t_2 \in [0, T]$,

$$(2.6) \qquad |P\left[a(t_1)\right] - P\left[a(t_2)\right]| \leq \frac{|K(t_1) - K(t_2)|}{N(t_2)} + \frac{|K(t_1)\left(N(t_1) - N(t_2)\right)|}{N(t_1)N(t_2)},$$

where

$$K(t) = 2F_0(t) + \sum_{k=1}^{\infty} \frac{2}{\pi k} F_{2k}(t) - E'(t),$$

$$N(t) = \sum_{k=1}^{\infty} 8\pi k \left(\varphi_{2k} e^{-(2\pi k)^2 \int\limits_0^t a(s)ds} + \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 \int\limits_\tau^t a(s)ds} d\tau \right).$$

Using the estimate

$$\left| e^{-(2\pi k)^2 \int_0^{t_1} a(s)ds} - e^{-(2\pi k)^2 \int_0^{t_2} a(s)ds} \right| \le (2\pi k)^2 \, |t_1 - t_2| \max_{[0,T]} a(t),$$

we obtain

$$(2.7) \qquad |N(t_1) - N(t_2)| \le \left[(C_4 + C_5) \frac{C_1}{C_2} + C_6 \right] |t_1 - t_2|,$$

where

$$C_4 = \sum_{k=1}^{\infty} 4 \, (2\pi k)^3 \, \varphi_{2k}, \; C_5 = \int_0^T \sum_{k=1}^{\infty} 4 \, (2\pi k)^3 \, F_{2k}(\tau) d\tau, \; C_6 = \max_{t \in [0,T]} \left(\sum_{k=1}^{\infty} 8\pi k F_{2k}(t) \right).$$

To this end, take an arbitrary $\varepsilon > 0$.

Since $K(t)$ is continuous in $[0, T]$, then $\exists \delta_1 = \delta_1(\varepsilon), \forall t_1, t_2 \in [0, T] \, (|t_1 - t_2| < \delta_1) :$

$$(2.8) \qquad |K(t_1) - K(t_2)| < \frac{C_2 \varepsilon}{2}.$$

Let

$$\delta = \min \left\{ \delta_1(\varepsilon), \; \frac{C_2^3}{2 \left((C_4 + C_5) \, C_1 + C_2 C_6 \right) C_1} \varepsilon \right\}.$$

From (2.7) for $|t_1 - t_2| < \delta$, we obtain

$$(2.9) \qquad |N(t_1) - N(t_2)| \le \frac{C_2^2}{2C_1} \varepsilon.$$

Substituting (2.8) and (2.9) in (2.6) we get

$$|P\,[a(t_1)] - P\,[a(t_2)]| < \epsilon.$$

So, the set $P(M_1)$ is equicontinuous. Then $P(M_1)$ is a compact set and the operator P is compact and maps the set M onto itself. Employing the Schauder's Fixed Point Theorem, we have a solution $a(t) \in C\,[0, T]$ of the equation (2.4).

Now let us show that there exists $Q_{T_0} \; (0 < T_0 \le T)$ and the solution (a, u) of the problem (0.1)-(0.4) is unique in Q_{T_0}. Suppose that (b, v) is also a solution pair of the problem (0.1)-(0.4). Then from the representation (2.2) and (2.4) of the solution, we have

$$(2.10) \quad u(x,t) - v(x,t) = \sum_{k=1}^{\infty} \varphi_{2k} \left(e^{-(2\pi k)^2 \int_0^t a(s)ds} - e^{-(2\pi k)^2 \int_0^t b(s)ds} \right) X_{2k}(x)$$

$$+ \sum_{k=1}^{\infty} \left(\int_0^t F_{2k}(\tau) \left(e^{-(2\pi k)^2 \int_\tau^t a(s)ds} - e^{-(2\pi k)^2 \int_\tau^t b(s)ds} \right) d\tau \right) X_{2k}(x)$$

$$+ \sum_{k=1}^{\infty} \left(\varphi_{2k-1} - 4\pi k \varphi_{2k} t \right) \left(e^{-(2\pi k)^2 \int_0^t a(s)ds} - e^{-(2\pi k)^2 \int_0^t b(s)ds} \right) X_{2k-1}(x)$$

$$+\sum_{k=1}^{\infty}\left(\int_0^t (F_{2k-1}(\tau)-4\pi k F_{2k}(\tau)(t-\tau))\left(e^{-(2\pi k)^2\int_\tau^t a(s)ds}-e^{-(2\pi k)^2\int_\tau^t b(s)ds}\right)d\tau\right)X_{2k-1}(x),$$

$$(2.11) \qquad a(t)-b(t)=P\left[a(t)\right]-P\left[b(t)\right],$$

where

$$P\left[a(t)\right]-P\left[b(t)\right] \quad = \quad \frac{2F_0(t)+\sum\limits_{k=1}^{\infty}\frac{2}{\pi k}F_{2k}(t)-E'(t)}{\sum\limits_{k=1}^{\infty}8\pi k\left(\varphi_{2k}e^{-(2\pi k)^2\int_0^t a(s)ds}+\int_0^t F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^t a(s)ds}d\tau\right)}-$$

$$\frac{2F_0(t)+\sum\limits_{k=1}^{\infty}\frac{2}{\pi k}F_{2k}(t)-E'(t)}{\sum\limits_{k=1}^{\infty}8\pi k\left(\varphi_{2k}e^{-(2\pi k)^2\int_0^t b(s)ds}+\int_0^t F_{2k}(\tau)e^{-(2\pi k)^2\int_\tau^t b(s)ds}d\tau\right)}.$$

The following estimate is true:

$$|P\left[a(t)\right]-P\left[b(t)\right]| \quad \le \quad \frac{\left(2F_0(t)+\sum\limits_{k=1}^{\infty}\frac{2}{\pi k}F_{2k}(t)+|E'(t)|\right)}{C_2^2}\cdot$$

$$\left(\sum_{k=1}^{\infty}8\pi k\varphi_{2k}\left|e^{-(2\pi k)^2\int_0^t a(s)ds}-e^{-(2\pi k)^2\int_0^t b(s)ds}\right| +\sum_{k=1}^{\infty}8\pi k\int_0^T F_{2k}(\tau)\left|e^{-(2\pi k)^2\int_\tau^t a(s)ds}-e^{-(2\pi k)^2\int_\tau^t b(s)ds}\right|d\tau\right).$$

Using the estimates

$$\left|e^{-(2\pi k)^2\int_0^t a(s)ds}-e^{-(2\pi k)^2\int_0^t b(s)ds}\right| \quad \le \quad (2\pi k)^2 T\max_{0\le t\le T}|a(t)-b(t)|,$$

$$\left|e^{-(2\pi k)^2\int_\tau^t a(s)ds}-e^{-(2\pi k)^2\int_\tau^t b(s)ds}\right| \quad \le \quad (2\pi k)^2 T\max_{0\le t\le T}|a(t)-b(t)|,$$

we obtain

$$\max_{0\le t\le T}|P\left[a(t)\right]-P\left[b(t)\right]|\le\alpha\max_{0\le t\le T}|a(t)-b(t)|.$$

Let $\alpha\in(0,1)$ be arbitrary fixed number. Fix a number T_0, $0<T_0\le T$, such that

$$\frac{C_1(C_4+C_5)}{C_2^2}T_0\le\alpha.$$

Then from the equality (2.11) we obtain

$$\|a - b\|_{C[0,T_0]} \leq \alpha \|a - b\|_{C[0,T_0]},$$

which implies that $a = b$. By substituting $a = b$ in (2.10), we have $u = v$. □

Theorem has been proved.

3. Continuous Dependence of (a, u) upon the data

Theorem 2. *Under assumption* $(A_1) - (A_3)$, *the solution* (a, u) *of the problem (0.1)-(0.3) depends continuously upon the data for small T .*

Proof. Let $\Phi = \{\varphi, F, E\}$ and $\overline{\Phi} = \{\overline{\varphi}, \overline{F}, \overline{E}\}$ be two sets of the data, which satisfy the conditions $(A_1) - (A_3)$. Let us denote $\|\Phi\| = (\|\varphi\|_{C^4[0,1]} + \|F\|_{C^{4,0}(\overline{Q}_T)} + \|E\|_{C^1[0,T]})$. Suppose that there exist positive constants M_i, $i = 1, 2, 3$ such that

$$\|\varphi\|_{C^4[0,1]} \leq M_1, \ \|F\|_{C^{4,0}(\overline{Q_T})} \leq M_2, \ \|E\|_{C^1[0,T]} \leq M_3,$$

$$\|\overline{\varphi}\|_{C^4[0,1]} \leq M_1, \ \|\overline{F}\|_{C^{4,0}(\overline{Q_T})} \leq M_2, \ \|\overline{E}\|_{C^1[0,T]} \leq M_3.$$

Let (a, u) and $(\overline{a}, \overline{u})$ be solutions of the inverse problem (0.1)-(0.4) corresponding to the data Φ and $\overline{\Phi}$, respectively. According to (2.4),

$$a(t) = \frac{2F_0(t) + \sum\limits_{k=1}^{\infty} \frac{2}{\pi k} F_{2k}(t) - E'(t)}{\sum\limits_{k=1}^{\infty} 8\pi k \left(\varphi_{2k} e^{-(2\pi k)^2 \int\limits_0^t a(s)ds} + \int\limits_0^t F_{2k}(\tau) e^{-(2\pi k)^2 \int\limits_\tau^t a(s)ds} d\tau \right)},$$

$$\overline{a}(t) = \frac{2\overline{F}_0(t) + \sum\limits_{k=1}^{\infty} \frac{2}{\pi k} \overline{F}_{2k}(t) - \overline{E}'(t)}{\sum\limits_{k=1}^{\infty} 8\pi k \left(\overline{\varphi}_{2k} e^{-(2\pi k)^2 \int\limits_0^t \overline{a}(s)ds} + \int\limits_0^t \overline{F}_{2k}(\tau) e^{-(2\pi k)^2 \int\limits_\tau^t \overline{a}(s)ds} d\tau \right)}.$$

First let us estimate the difference $a - \overline{a}$. It is easy to compute that

$$\left| F_0(t) \sum_{k=1}^{\infty} 8\pi k \overline{\varphi}_{2k} e^{-(2\pi k)^2 \int\limits_0^t \overline{a}(s)ds} - \overline{F}_0(t) \sum_{k=1}^{\infty} 8\pi k \varphi_{2k} e^{-(2\pi k)^2 \int\limits_0^t a(s)ds} \right|$$

$$\leq \ M_4 \|a - \overline{a}\|_{C[0,T]} + M_5 \|\varphi - \overline{\varphi}\|_{C^4[0,1]} + M_6 \|F - \overline{F}\|_{C^{4,0}(\overline{Q}_T)},$$

$$\left| F_0(t) \sum_{k=1}^{\infty} 8\pi k \int\limits_0^t \overline{F}_{2k}(\tau) e^{-(2\pi k)^2 \int\limits_\tau^t \overline{a}(s)ds} d\tau - \overline{F}_0(t) \sum_{k=1}^{\infty} 8\pi k \int\limits_0^t F_{2k}(\tau) e^{-(2\pi k)^2 \int\limits_\tau^t a(s)ds} d\tau \right|$$

$$\leq \ M_7 \|a - \overline{a}\|_{C[0,T]} + 2T M_5 \|F - \overline{F}\|_{C^{4,0}(\overline{Q}_T)},$$

$$\left| \sum_{k=1}^{\infty} \frac{2}{\pi k} F_{2k}(t) \sum_{k=1}^{\infty} 8\pi k \overline{\varphi}_{2k} e^{-(2\pi k)^2 \int_0^t \overline{a}(s)ds} - \sum_{k=1}^{\infty} \frac{2}{\pi k} \overline{F}_{2k}(t) \sum_{k=1}^{\infty} 8\pi k \varphi_{2k} e^{-(2\pi k)^2 \int_0^t a(s)ds} \right|$$

$$\leq \frac{M_4}{6} \left\| a - \overline{a} \right\|_{C[0,T]} + \frac{M_5}{6} \left\| \varphi - \overline{\varphi} \right\|_{C^4[0,1]} + \frac{M_6}{6} \left\| F - \overline{F} \right\|_{C^{4,0}(\overline{Q}_T)},$$

$$\left| \sum_{k=1}^{\infty} \frac{2}{\pi k} F_{2k}(t) \sum_{k=1}^{\infty} 8\pi k \int_0^t \overline{F}_{2k}(\tau) e^{-(2\pi k)^2 \int_\tau^t \overline{a}(s)ds} d\tau \right.$$

$$\left. - \sum_{k=1}^{\infty} \frac{2}{\pi k} \overline{F}_{2k}(t) \sum_{k=1}^{\infty} 8\pi k \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 \int_\tau^t a(s)ds} d\tau \right|$$

$$\leq \frac{T^2 M_7}{6} \left\| a - \overline{a} \right\|_{C[0,T]} + \frac{T M_5}{3} \left\| F - \overline{F} \right\|_{C^{4,0}(\overline{Q}_T)},$$

$$\left| E'(t) \sum_{k=1}^{\infty} 8\pi k \overline{\varphi}_{2k} e^{-(2\pi k)^2 \int_0^t \overline{a}(s)ds} - \overline{E}'(t) \sum_{k=1}^{\infty} 8\pi k \varphi_{2k} e^{-(2\pi k)^2 \int_0^t a(s)ds} \right|$$

$$\leq M_8 \left\| a - \overline{a} \right\|_{C[0,T]} + M_9 \left\| \varphi - \overline{\varphi} \right\|_{C^4[0,1]} + M_6 \left\| E - \overline{E} \right\|_{C^1[0,T]},$$

$$\left| E'(t) \sum_{k=1}^{\infty} 8\pi k \int_0^t \overline{F}_{2k}(\tau) e^{-(2\pi k)^2 \int_\tau^t \overline{a}(s)ds} d\tau - \overline{E}'(t) \sum_{k=1}^{\infty} 8\pi k \int_0^t F_{2k}(\tau) e^{-(2\pi k)^2 \int_\tau^t a(s)ds} d\tau \right|$$

$$\leq M_{10} \left\| a - \overline{a} \right\|_{C[0,T]} + T M_9 \left\| F - \overline{F} \right\|_{C^{4,0}(\overline{Q}_T)} + T M_5 \left\| E - \overline{E} \right\|_{C^1[0,T]},$$

where M_k, $k = 4, ..., 12$, are some constants.

If we consider these estimates in $a - \overline{a}$, we obtain

$$(1 - M_{11}) \left\| a - \overline{a} \right\|_{C[0,T]} \leq M_{12} \left(\left\| E - \overline{E} \right\|_{C^1[0,T]} + \left\| \varphi - \overline{\varphi} \right\|_{C^4[0,1]} + \left\| F - \overline{F} \right\|_{C^{4,0}(\overline{Q}_T)} \right).$$

The inequality $M_{11} < 1$ holds for small T. Finally, we obtain

$$\left\| a - \overline{a} \right\|_{C[0,T]} \leq M_{13} \left\| \Phi - \overline{\Phi} \right\|, \quad M_{13} = \frac{M_{12}}{(1 - M_{11})}.$$

From (2.2), the similar estimate is also obtained for the difference $u - \overline{u}$ as

$$\left\| u - \overline{u} \right\|_{C(\overline{Q}_T)} \leq M_{14} \left\| \Phi - \overline{\Phi} \right\|.$$

$\square$

4. NUMERICAL METHOD AND AN EXAMPLE

We consider an example of numerical solution of the inverse problem (0.1)-(0.4). We use the finite difference method with a predictor-corrector type approach, that is suggested in [2]. Apply this method to the problem (0.1)-(0.4).

We subdivide the intervals $[0, 1]$ and $[0, T]$ into M and N subintervals of equal lengths $h = \frac{1}{M}$ and $\tau = \frac{T}{N}$ respectively. Then we add a line $x = (M + 1)h$ to

generate the fictitious point needed for dealing with the second boundary condition. We choose the Crank-Nicolson scheme, which is absolutely stable and has a second order accuracy in both h and τ. ([9]) The Crank-Nicolson scheme for (0.1)-(0.4) is as follows:

$$\frac{1}{\tau}\left(u_i^{j+1} - u_i^j\right) = \frac{1}{2}\left(a^{j+1} + a^j\right)\frac{1}{2h^2}\left[\left(u_{i-1}^j - 2u_i^j + u_{i+1}^j\right) + \left(u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}\right)\right]$$

$$(4.1) \qquad + \frac{1}{2}\left(F_i^{j+1} + F_i^j\right),$$

$$(4.2) \qquad\qquad\qquad u_i^0 = \phi_i,$$

$$(4.3) \qquad\qquad\qquad u_0^j = u_M^j,$$

$$(4.4) \qquad\qquad\qquad u_{M-1}^j = u_{M+1}^j,$$

where $1 \le i \le M$ and $0 \le j \le N$ are the indices for the spatial and time steps respectively, $u_i^j = u(x_i, t_j)$, $\phi_i = \varphi(x_i)$, $F_i^j = F(x_i, t_j)$, $x_i = ih$, $t_j = j\tau$. At the $t = 0$ level, adjustment should be made according to the initial condition and the compatability requirements.

(4.1)-(4.4) problem forms $M \times M$ linear system of equations

$$(4.5) \qquad\qquad\qquad AU^{j+1} = b,$$

where
$$U^j = \left(u_1^j, u_2^j, ..., u_M^j\right)^T, 1 \le j \le N, b = (b_1, b_2, ..., b_M)^T,$$

$$A = \begin{bmatrix} -2(1+R) & 1 & 0 & ... & & 0 & 1 \\ 1 & -2(1+R) & 1 & 0 & ... & & 0 \\ 0 & 1 & -2(1+R) & 1 & 0 & ... & 0 \\ \vdots & & & \ddots & & & \\ & & & & 0 & 1 & -2(1+R) & 1 \\ & & & & & 0 & 2 & -2(1+R) \end{bmatrix},$$

$R = \frac{2h^2}{\tau(a^{j+1}+a^j)}, j = 0, 1, ..., N,$
$b_1 = 2(1 - R)u_1^j - u_2^j - u_M^j - R\tau(F_1^{j+1} + F_1^j), j = 0, 1, ..., N,$
$b_M = -2u_{M-1}^j + 2(1 - R)u_M^j - R\tau(F_M^{j+1} + F_M^j), j = 0, 1, ..., N,$
$b_i = -u_{i-1}^j + 2(1-R)u_i^j - u_{i+1}^j - R\tau(F_i^{j+1} + F_i^j), i = 2, 3, ..., M-1, j = 0, 1, ..., N.$

Now, let us construct the predicting-correcting mechanism. First, integrating the equation (0.1) respect to x from 0 to 1 and using (0.3),(0.4), we obtain

$$(4.6) \qquad\qquad a(t) = \frac{-E'(t) + \int_0^1 F(x,t)dx}{u_x(0,t)}.$$

The finite difference approximation of (4.6) is

$$a^j = \frac{\left(-(Et)^j + (Fin)^j\right)h}{u_1^j - u_0^j},$$

where $(Et)^j = E'(t_j)$, $(Fin)^j = \int_0^1 F(x, t_j)dx$, $j = 0, 1, ..., N$. For $j = 0$,

$$a^0 = \frac{\left(-(Et)^0 + (Fin)^0\right)h}{\phi_1 - \phi_0},$$

and the values of ϕ_i provide us to start our computation. We denote the values of a^j, u_i^j at the s-th iteration step $a^{j(s)}$, $u_i^{j(s)}$, respectively. In numerical computation, since the time step is very small, we can take $a^{j+1(0)} = a^j$, $u_i^{j+1(0)} = u_i^j$, $j = 0, 1, 2, ...N$, $i = 1, 2, ..., M$. At each $(s+1)$-th iteration step we first determine $a^{j+1(s+1)}$ from the formula

$$a^{j+1(s+1)} = \frac{\left(-(Et)^{j+1} + (Fin)^{j+1}\right)h}{u_1^{j+1(s)} - u_0^{j+1(s)}}.$$

Then from (4.1)-(4.3) we obtain

$$\frac{1}{\tau}\left(u_i^{j+1(s+1)} - u_i^{j+1(s)}\right) = \frac{1}{4h^2}\left(a^{j+1(s+1)} + a^{j+1(s)}\right)\left[\left(u_{i-1}^{j+1(s)} - 2u_i^{j+1(s)} + u_{i+1}^{j+1(s)}\right)\right.$$

$$(4.7) \qquad \left. + \left(u_{i-1}^{j+1(s+1)} - 2u_i^{j+1(s+1)} + u_{i+1}^{j+1(s+1)}\right)\right] + \frac{1}{2}\left(F_i^{j+1} + F_i^j\right),$$

$$(4.8) \qquad\qquad u_0^{j+1(s)} = u_M^{j+1(s)},$$

$$(4.9) \qquad\qquad u_{M-1}^{j+1(s)} = u_{M+1}^{j+1(s)}, \quad s = 0, 1, 2, ... \; .$$

The problem (4.7)-(4.9) can be solved by the Gauss elimination method and $u_i^{j+1(s+1)}$ is determined. If the difference of values on two iteration reaches the prescribed tolerence, the iteration is stopped and we accept the corresponding values $a^{j+1(s+1)}$, $u_i^{j+1(s+1)}(i = 1, 2, ..., M)$ as a^{j+1}, $u_i^{j+1}(i = 1, 2, ..., M)$, on the $(j+1)$-th time step, respectively. In virtue of this iteration, we can move from level j to level $j + 1$.

Example. Consider the inverse problem (0.1)-(0.4), with

$$F(x, t) = \left(\frac{1}{\pi}\exp(-t) + 4\pi\exp(3t)\right)\cos 2\pi x + (2\pi)^2(1 - x)\sin 2\pi x \exp(3t),$$

$$\varphi(x) = (1 - x)\sin 2\pi x, \quad E(t) = \frac{1}{2\pi}\exp(-t), \quad T = \frac{1}{4}.$$

It is easy to check that the exact solution is

$$\{a(t), \; u(x, t)\} = \left\{\frac{1}{(2\pi)^2} + \exp(4t), \; (1 - x)\sin 2\pi x \exp(-t)\right\}.$$

We use the Crank-Nicolson scheme and the iteration which are explained above. In result, we obtain Table 1 and Table 2 for exact and approximate values of $a(t)$ and $u(x, t)$. The step sizes are $h = 0.005$ and $\tau = \frac{h}{4}$.

References

[1] J. R. Cannon, Y. Lin, S.Wang, Determination of a control parameter in a parabolic partial differential equation, J. Austral. Math. Soc. Ser. B. 33 (1991) 149-163.

[2] J. R. Cannon, Y. Lin, S.Wang, Determination of source parameter in a parabolic equations, Meccanica 27 (2) (1992) 85-94.

[3] A. G. Fatullayev, N. Gasilov, I.Yusubov, Simultaneous determination of unknown coefficients in a parabolic equation, Applicable Analysis 87 (10-11) (2008) 1167-1177.

[4] M. I. Ivanchov, Inverse problems for the heat-conduction equation with nonlocal boundary condition, Ukrainian Mathematical Journal 45 (8) (1993) 1186-1192.

[5] M. I. Ivanchov, N. V. Pabyrivska, Simultaneous determination of two coefficients of a parabolic equation in the case of nonlocal and integral conditions, Ukrainian Mathematical Journal 53 (5) (2001) 674-684.

[6] W. Liao, M. Dehghan, A. Mohebbi, Direct numerical method for an inverse problem of a parabolic partial differential equation, Journal of Computational and Applied Mathematics 232 (2009) 351-360.

[7] N.I. Ionkin, Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition, Differential Equations 13 (1977) 204-211.

[8] V. A. Il'in, Necessary and sufficient conditions for the subsystem of eigenfunctions and associated functions of Keldysh's pencil of ordinary differential operators to form the basis, Soviet Math. Dokl. 17 (1976) 513-516.

[9] A.A. Samarskii, The Theory of Difference Schemes, Marcel Dekker, Inc., New York, 2001.

Table 1. The some values of $a(t)$

Exact	Approximate	Error	Relative Error
1.0354	1.0261	0.0093	0.009
1.3685	1.3276	0.0409	0.0299
1.6658	1.6308	0.035	0.021
1.9698	1.9487	0.0211	0.0107
2.0798	2.0644	0.0154	0.0074
2.2068	2.2008	0.006	0.0027
2.3186	2.322	0.0034	0.0014
2.4727	2.4915	0.0188	0.0076
2.5981	2.6304	0.0323	0.0124
2.7301	2.7778	0.0477	0.0175

Table 2. The some values of $u(x,t)$ for $T = 70$

Exact	Approximate	Error	Relative Error
0.3633	0.3486	0.0147	0.0347
0.5952	0.5933	0.0019	0.0044
0.6506	0.6523	0.0017	0.0001
0.4054	0.4084	0.003	0.0017
0.3402	0.3415	0.0013	0.0016
0.2563	0.2554	0.0009	0.0085
-0.2157	-0.2298	0.0141	0.0457
-0.1430	-0.1517	0.0087	0.0066
-0.1176	-0.1252	0.0076	0.0019
-0.0754	-0.0812	0.0059	0.0276

*⁾Department of Mathematics, Gebze Institute of Technology,, Gebze-Kocaeli 41400, Turkey, Tel/Fax: +902626051641/+902626051365, **⁾Department of Mathematics, Kocaeli University, Kocaeli 41380, Turkey

E-mail address: mismailov@gyte.edu.tr; fzoroglu@kocaeli.edu.tr